Partial Fractions

Example

Express $\dfrac{x^2+x+1}{x^2-2x-3}$ as the sum of partial fractions.

Solution

Begin by factorising the denominator:

$$x^2-2x-3=(x-3)(x+1)$$

Now observe that the highest degree of both numerator and denominator is the same. This means that the partial fraction decomposition will look like

$$A+\frac{B}{x-3}+\frac{C}{x+1}$$

We first need to work out the whole number, A, as follows.

$$\begin{aligned}\frac{x^2+x+1}{x^2-2x-3} &= \frac{x^2-2x-3+2x+3+x+1}{x^2-2x-3}\\ &= \frac{(x^2-2x-3)+(3x+4)}{x^2-2x-3}\\ &= 1+\frac{3x+4}{x^2-2x-3}\end{aligned}$$

Now:

$$\begin{aligned}\frac{3x+4}{x^2-2x-3} &= \frac{3x+4}{(x-3)(x+1)} = \frac{B}{x-3}+\frac{C}{x+1}\\ &= \frac{B(x+1)+C(x-3)}{(x-3)(x+1)}\end{aligned}$$

Equating numerators:

$$3x+4=B(x+1)+C(x-3)$$

Since this is true for all values of x, solve for B and C by evaluating at any two values of x. Choose $x=-1$ and $x=3$.

$$\begin{aligned}x=-1: &\quad 1=-4C \quad \Longrightarrow C=-\tfrac{1}{4}\\ x=3: &\quad 13=4B \quad \Longrightarrow B=\tfrac{13}{4}\end{aligned}$$

Thus

$$\begin{aligned}\frac{x^2+x+1}{x^2-2x-3} &= 1-\frac{\frac{1}{4}}{x-3}+\frac{\frac{13}{4}}{x+1}\\ &= 1-\frac{1}{4(x-3)}+\frac{13}{4(x+1)}\end{aligned}$$