Second-order, nonhomogeneous Cauchy-Euler differential equation

Example

Solve

$$x^2y'' - xy' - 3y = 2x^3$$

Solution

This is a nonhomogeneous Cauchy-Euler equation as it has the form

$$ax^2y'' + bxy' + cy = r(x)$$

Begin by finding the homogeneous solution. That is, solve

$$x^2y'' - xy' - 3y = 0 \tag{1}$$

For Cauchy-Euler equations we guess a solution of

$$y = x^m$$

So:

$$y' = mx^{m-1}$$

and

$$y'' = m(m-1)x^{m-2}$$

Putting these values into Equation (1) give us

$$m(m-1)x^m - mx^m - 3x^m = 0$$

Factoring out the x^m term leaves us with

$$\begin{aligned} m(m-1) - m - 3 &= 0 \\ m^2 - 2m - 3 &= 0 \\ (m-3)(m+1) &= 0 \\ m &= 3 \text{ or } m = -1 \end{aligned}$$

This equation has two distinct real roots, so the homogeneous solution has the form

$$y_h(x) = c_1x^{-1} + c_2x^3$$

Next we need to find a particular solution, $y_p(x)$. We can do this using the method of Variation of Parameters. A particular solution is given by

$$y_p(x) = \left[\int \frac{-y_2r(x)}{W}dx\right] y_1 + \left[\int \frac{y_1r(x)}{W}dx\right] y_2$$

where $y_1 = x^{-1}$ and $y_2 = x^3$ To find $r(x)$ we need to make the coefficient of y'' equal to 1 in the original equation by dividing through by x^2. That is

$$y'' - \frac{1}{x}y' - \frac{3}{x^2} = 2x$$

So $r(x) = 2x$.

$$\begin{aligned}
W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{-1} & x^3 \\ -x^{-2} & 3x^2 \end{vmatrix} = x^{-1}.3x^2 - x^3.(-x^{-2}) = 4x \\
y_p &= \left[\int \frac{-y_2 r(x)}{W}dx\right] y_1 + \left[\int \frac{y_1 r(x)}{W}dx\right] y_2 \\
&= \left[\int \frac{-x^3.2x}{4x}dx\right].x^{-1} + \left[\int \frac{x^{-1}2x}{4x}dx\right].x^3 \\
&= -\frac{1}{2x}\int x^3 dx + \frac{x^3}{4}\int \frac{1}{x}dx \\
&= -\frac{x^3}{8} + \frac{x^3}{4}\ln|x|
\end{aligned}$$

The general solution is given by the sum of the homogeneous solution and the particular solution:

$$y(x) = y_h(x) + y_p(x)$$

$$\therefore y(x) = c_1 x^{-1} + c_2 x^3 - \frac{x^3}{8} + \frac{x^3}{4}\ln|x|$$