Integral Calculus

3.0 Objectives

This chapter covers the following topics related to integral calculus. After successful completion of this section, you will be able to:

- Evaluate indefinite and definite integrals.
- Use the substitution method and integration-by-parts to evaluate integrals.
- Integration of trigonometric and hyperbolic functions.
- Derive reduction formulae and use these to evaluate integrals.
- Integrate using other methods, such as the method of partial fractions and completing the square.
- Evaluate the area under the curve and between the curves.
- Find the volume of a solid of revolution.
- Determine the lengths of plane curves.
- Find the centre of mass.
- Determine the MacLaurin series expansion for some common functions.
- Define power series and use power series to evaluate integrals.

3.1 Indefinite Integrals

i) The video in the link below demonstrates how to apply the indefinite integral for the following trigonometric functions.

a) $\int \cos x \, dx$ b) $\int \sin x \, dx$ c) $\int \sec^2 x \, dx$

d) $\int \frac{3}{7} \sin x \, dx$ e) $\int \frac{4 \sec^2 x}{5} \, dx$ f) $\int \frac{5}{\sec x} \, dx$

g) $\int \frac{2}{7} \sin x \, dx$ h) $\int \frac{3}{8 \csc x} \, dx$ i) $\int \frac{-4 \cos x}{7} \, dx$

j) $\int 3x - \frac{\sec^2 x}{8} \, dx$ k) $\int 1 + \tan^2 x \, dx$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/M0WcSCxYl2Y`

ii) The video in the link below demonstrates how to apply the indefinite integral for the following exponential functions.

a) $\displaystyle\int \frac{2}{7}e^{x}\,dx$ b) $\displaystyle\int \frac{3}{5}e^{2x}\,dx$ c) $\displaystyle\int \frac{4}{3e^{5x}}\,dx$

d) $\displaystyle\int \frac{2}{3}e^{3x-2}\,dx$ e) $\displaystyle\int 3e^{7x}-\frac{2}{3}e^{5x-1}\,dx$ f) $\displaystyle\int 3e^{5-2x}-\frac{3}{2e^{4x}}\,dx$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/S8-nsBjcnF8`

3.2 Definite Integration

i) The video in the link below explains what a definite integral is and demonstrates how to apply it to the following functions.

a) $\displaystyle\int_1^3 3x^2\,dx$ b) $\displaystyle\int_{-1}^2 (4x^3-3)\,dx$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/6AS2qyKbbNI`

ii) The video below further demonstrates how to find the definite integral for algebraic expressions with various indices, including trigonometric functions.

a) $\displaystyle\int_2^4 x^3\,dx$ b) $\displaystyle\int_4^{10} 7\,dx$ c) $\displaystyle\int_1^2 (3x^2-5x+2)\,dx$

d) $\displaystyle\int_{-1}^3 (2x+3)^2\,dx$ e) $\displaystyle\int_{\frac{1}{2}}^1 \frac{1}{x^2}\,dx$ f) $\displaystyle\int_1^e \frac{1}{x}\,dx$

g) $\displaystyle\int_4^9 \sqrt{x}\,dx$ h) $\displaystyle\int_2^3 \frac{x^3-5x^2}{x}\,dx$ i) $\displaystyle\int_0^{\frac{1}{3}} e^{3x}\,dx$

j) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x\,dx$ k) $\displaystyle\int_0^{\frac{\pi}{4}} \sin(2x)\,dx$ l) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sec^2 x\,dx$

m) $\displaystyle\int_0^1 x^2(x^3+5)^2\,dx$ n) $\displaystyle\int_1^2 4xe^{x^2}\,dx$ o) $\displaystyle\int_0^1 xe^{x}\,dx$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/rCWOdfQ3cwQ

3.3 Integration techniques

The following section provides worked examples of integration by substitution and integration by parts techniques for both definite and indefinite integrals.

i) Integration by substitution: Indefinite integrals

a) $\displaystyle\int 4x(x^2+5)^3\,dx$ b) $\displaystyle\int 8\cos(4x)\,dx$ c) $\displaystyle\int x^3e^{x^4}\,dx$

d) $\displaystyle\int 8\sqrt{40-2x^2}\,dx$ e) $\displaystyle\int \frac{x^3}{(2+x^4)^2}\,dx$ f) $\displaystyle\int \sin^4 x\cos x\,dx$

g) $\displaystyle\int \sqrt{5x+4}\,dx$ h) $\displaystyle\int x\sqrt{3x+2}\,dx$ i) $\displaystyle\int 2x\sqrt{4x-5}\,dx$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/sdYdnpYn-1o

ii) Integration by substitution: Definite integrals

a) $\displaystyle\int_0^2 2x(x^2+4)^2\,dx$ b) $\displaystyle\int_0^4 4x\sqrt{16-x^2}\,dx$ c) $\displaystyle\int_1^2 \frac{2x}{(1+x^2)^3}\,dx$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/tM4RWc9ryx0

iii) Integration by parts: Indefinite integrals

a) $\displaystyle\int xe^x \, dx$ b) $\displaystyle\int x \sin x \, dx$ c) $\displaystyle\int x^2 \ln x \, dx$

d) $\displaystyle\int \ln x \, dx$ e) $\displaystyle\int x^2 \sin x \, dx$ f) $\displaystyle\int x \cos x \, dx$

g) $\displaystyle\int x^2 e^x \, dx$ h) $\displaystyle\int (\ln x)^2 \, dx$ i) $\displaystyle\int \ln x^7 \, dx$

j) $\displaystyle\int e^x \sin x \, dx$ k) $\displaystyle\int \frac{(\ln x)^2}{x} \, dx$ l) $\displaystyle\int e^{3x} \cos(4x) \, dx$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/sWSLLO3DS1I`

iv) Integration by parts: Definite integrals

a) $\displaystyle\int_1^e x^2 \ln x \, dx$ b) $\displaystyle\int_0^1 x^2 e^x \, dx$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/E6VLoKoJWXY`

3.4 Integration of Trigonometric and Hyperbolic Functions

The videos in this section demonstrate how to integrate trigonometric, inverse trigonometric, and hyperbolic functions using various techniques, including identities and standard formulas.

i) Integration using trigonometric identities

a) $\displaystyle\int \frac{3}{7} \sin x \, dx$ b) $\displaystyle\int \frac{4 \sec^2 x}{5} \, dx$ c) $\displaystyle\int \frac{5}{\sec x} \, dx$

d) $\displaystyle\int \frac{2}{7} \sin x \, dx$ e) $\displaystyle\int \frac{3}{8 \csc x} \, dx$ f) $\displaystyle\int \frac{-4 \cos x}{7} \, dx$

g) $\displaystyle\int \left(3x - \frac{\sec^2 x}{8}\right) dx$ h) $\displaystyle\int \left(1 + \tan^2 x\right) dx$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/M0WcSCxYl2Y

a) $\displaystyle\int \frac{3}{4}\sin(5x-2)\,dx$ b) $\displaystyle\int \frac{4\sec^2(2-3x)}{5}\,dx$ c) $\displaystyle\int \frac{3}{\sec 2x}\,dx$

d) $\displaystyle\int 5\cos(4x-7)\,dx$ e) $\displaystyle\int \frac{2\sin(3-8x)}{7}\,dx$ f) $\displaystyle\int \frac{3}{\cos^2(2x-5)}\,dx$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/vRBLjaFlCMk

a) $\displaystyle\int \left(1+\tan^2 x\right)\,dx$ b) $\displaystyle\int \left(1+\tan^2(5\theta)\right)\,d\theta$ $\displaystyle\int \left(3+3\tan^2 2x\right)\,dx$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/3RIxvN-7IFI

a) $\displaystyle\int \sin x\cos x\,dx$ b) $\displaystyle\int 5\cos\left(\frac{3}{2}x\right)\sin\left(\frac{3}{2}x\right)\,dx$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/yc8Kepa1s20

a) $\displaystyle\int \sin^2 x\,dx$ b) $\displaystyle\int 3\sin^2(5\theta)\,d\theta$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/KrWPb6JTVRc

ii) Integration using inverse trigonometric functions

a) $\displaystyle\int \frac{2}{\sqrt{16-x^2}}\,dx$ **b)** $\displaystyle\int \frac{dx}{5+16x^2}$

c) $\displaystyle\int \frac{dx}{x\sqrt{9x^2-4}}$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/J6MvP8SYVfo`

iii) Integration by subsitution of hyperbolic functions

Set 1

a) $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$ b) $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$ c) $\displaystyle\int \frac{1}{25-x^2}\,dx$

Set 2

a) $\displaystyle\int \frac{1}{\sqrt{3x^2+27}}\,dx$ b) $\displaystyle\int \frac{1}{\sqrt{x^2-6x}}\,dx$

c) $\displaystyle\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$ d) $\displaystyle\int \frac{1}{\sqrt{12x+2x^2}}\,dx$

Set 3

a) $\displaystyle\int \sqrt{1+x^2}\,dx$ b) $\displaystyle\int_0^6 \frac{x^3}{\sqrt{x^2+9}}\,dx$ c) $\displaystyle\int \frac{1}{\sqrt{4x^2-12x-7}}\,dx$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/jOMwIMtrUeQ`

3.5 Reduction Formulae

The following link demonstrates how to use the reduction formula method to evaluate the integrals listed below.

a) $\displaystyle\int \cos^n x\,dx$ **b)** $\displaystyle\int_0^{\frac{\pi}{2}} x^n \sin x\,dx$

c) $\displaystyle\int \frac{x^n}{\sqrt{x+1}}\,dx$ **d)** $\displaystyle\int x(\ln x)^{2n}\,dx$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/an8vp2QhONE

3.6 Integration by using partial fractions and completing the square

i) The link below demonstrates how to find the partial fractions for the following improper rational functions, but **not the integral**.

a) $\dfrac{5x-3}{x^2-3x-4}$ b) $\dfrac{6x-22}{2x^2+7x-15}$ c) $\dfrac{7x-11}{(x-2)^2}$

d) $\dfrac{3x^2-24x+53}{x^3-6x^2+9x}$ e) $\dfrac{6x^2+21x+11}{(x^2+3)(x+5)}$ f) $\dfrac{3x^2+5x-4}{(x^2-7)(x+1)}$

g) $\dfrac{3x^4-2x^3+6x^2-3x+3}{(x^2+2)^2(x+3)}$ h) $\dfrac{x^3+3}{x^2-2x-3}$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/2bMndF92Oqo

ii) The link below demonstrates how to evaluate integrals of the following rational functions by using partial fractions.

a) $\displaystyle\int \frac{1}{x^2-4}\,dx$ **b)** $\displaystyle\int \frac{x-4}{x^2+2x-15}\,dx$

c) $\displaystyle\int \frac{x}{(x-1)(x-2)^2}\,dx$ **d)** $\displaystyle\int \frac{x^2+9}{(x^2-1)(x^2+4)}\,dx$

Video Visit the URL below to view a video:

https://www.youtube.com/embed/6rXByMcuAyI

iii) The video in the link below demonstrates how to evaluate integrals using the completing the square method for the following questions.

a) $\displaystyle\int \frac{dx}{x^2 - 6x + 13}$ **b)** $\displaystyle\int \frac{x-5}{x^2 + 8x + 22}\,dx$

3.7 Evaluate the Area Under the Curve and Between the Curves

The video in the link below demonstrates how to find the area between a curve and the X-axis, between two curves along the X-axis, between a curve and the Y-axis and between two curves along the Y-axis.

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/kgg5Rspf1Js`

Then, it demonstrates how to evaluate the area of the region bounded by:

a) the line with equation $y = 8 - 2x$, the X-axis and the Y-axis.

b) the line with equation $y = x$ and the curve with equation $y = x^2$

c) the curves with equations $y = x^2$ and $x = y^2$

d) the curves with equations $x = 1 - y^2$ and $x = y^2 - 1$

e) the curve with equation $y = x^2 - 4x$ and the X-axis

f) the line with equation $y = 6 - 3x$ and the curve with equation $y = x^2 - 4x$

g) the line with equation $x = 3y - 2$ and the curve with equation $x = 2y^2 - 4$

3.8 Volume of a Solid of Revolution

The video at the link below demonstrates the methodology for finding the volume of revolution along the X- and Y-axes.

It also demonstrates evaluating the volume of the solid generated by the following curves between given values:

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/SAHSVg7Jw_A`

a) $y = \sqrt{x}$ rotating along the X-axis for $360°$ between $x = 0$ and $x = 4$

b) $y = \dfrac{1}{x}$ rotating along the X-axis for 360° between $x = 1$ and $x = 3$

c) $y = x^2$ rotating along the Y-axis for 360° between $y = 0$ and $y = 4$

d) $y = \dfrac{x^3}{2}$ rotating along the Y-axis for 360° between $x = 0$ and $y = 1$

3.9 The Length of a Curve

The video in the link below demonstrates how to evaluate the length of a curve for the following examples.

Video Visit the URL below to view a video:

https://www.youtube.com/embed/DNDAwWIL5FY

a) $f(x) = 1 + 6x^{\frac{3}{2}},\ 0 \leq x \leq 1$
b) $f(x) = \frac{3}{2}x^{\frac{2}{3}},\ 1 \leq x \leq 8$
c) $x = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}},\ 0 \leq y \leq 4$

3.10 Centre of Mass

The video that is comprised in the link below demonstrates the concept of the moment of a point mass about a point and extends the definition to a system of point masses about the origin. The centre of mass is then defined, and the physical significance is explained in terms of doors and see-saws.

Video Visit the URL below to view a video:

https://www.youtube.com/embed/ZxOIlCjh_pw

i) Moreover, the video also explains how to find the answer to the following question.

Two particles with masses 30 kg, 20 kg, and 50 kg are attached at points A, B, and C on a light, uniform rod.

The rod is supported at point $O(0,\ 0)$, acting as a pivot.

(a) Will the system be balanced about point O?

(b) If not, where should the fulcrum (pivot) be placed to achieve balance?

ii) The video in the link below demonstrates the method of finding the centre of mass for a non-uniform rod. It also demonstrates how to find the centre of mass of two examples that are listed below.

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/xxXUCYzdrlM`

(a) Find the centre of mass of a uniform rod of length L with constant density ρ.

(b) Find the centre of mass of a non-uniform rod of length L with density x.

iii) The video below demonstrates how to prove that the centre of mass of a uniform rod with mass M is at its middle.

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/HCIhzGG_ZVY`