2 Differential Calculus

2.0 Objectives

This chapter covers the following topics related to differential calculus. After successful completion of this section, you will be able to:

- Calculate the limit of a function.
- Determine whether a function is continuous or not at a given point.
- Define what is meant by the derivative of a function.
- Differentiate various functions using the appropriate techniques and rules.
- Use L'Hopital's Rule to find derivatives.
- Calculate the equation of a tangent line and a normal.
- Use differentials to estimate the change.
- Sketch curves using the first and second derivatives.
- Find vertical and horizontal asymptotes.
- Use differential calculus to solve related rate problems.

2.1 Introduction to Limits

This video provides an introduction to limits, helping you familiarise yourself with limits and how to find the limits for various functions.

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/riXcZT2ICjA`

2.2 Finding the Limit of a Given Function

This video demonstrates how to find the domain for the functions listed below.

a) $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$ b) $\lim_{x\to 2}(x^2+2x-4)$ c) $\lim_{x\to 3} \dfrac{x^3-27}{x-3}$

d) $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$ e) $\lim_{x\to 4} \dfrac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4}$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/YNstP0ESndU`

2.3 Checking the Continuity of a Function at a Given Point

This video demonstrates how to find the continuity check of a function at a given point. This process is explained using the 3-Step Continuity Test for the functions listed below.

$$a) f(x) = \begin{cases} \sqrt{x+2}, & x < 2 \\ x^2 - 2, & 2 \leq x < 3 \\ 2x + 5, & x \leq 3 \end{cases} \qquad b) f(x) = \begin{cases} 2x + 5, & x < -1 \\ x^2 + 2, & x > -1 \\ 5, & x = 1 \end{cases}$$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/WT7oxiiFYt8`

2.4 Derivative of a Function

This video demonstrates the concept of derivatives and how the limit approach is applied to find the derivative of a function, including the concept of rate of change, through specific examples.

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/wUZcIYx-7a4`

i) Derivative of polynomial functions

The video below shows how to find the derivative of the polynomial functions listed below.

a) $f(x) = x^3 - 5x^2 + 7x - 4$ b) $f(x) = 4x^5 - 6x^3 + 8x^2 - 9$

c) $f(x) = 7x(2x - x^3)$ d) $f(x) = (3x + 2)^2$

e) $f(x) = \dfrac{4x^5 - 5x^4 + 2x^3}{x^2}$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/8Sv6CNuNwqo`

ii) Derivative of implicit functions

This video demonstrates how to find the derivative of the implicit functions listed below.

a) $x^2 + y^2 = 100$ b) $x^3 + 4xy + y^2 = 13$ c) $5 - x^2 = \sin(xy^2)$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/xbviQHhU1rA`

iii) Derivative of exponential functions

The video below shows how to find the derivative of the exponential functions listed.

a) $\dfrac{d}{dx}\left[e^x\right]$ b) $\dfrac{d}{dx}\left[e^{5x+3}\right]$ c) $\dfrac{d}{dx}\left[e^{x^2}\right]$ d) $\dfrac{d}{dx}\left[e^{x^3+8x}\right]$

e) $\dfrac{d}{dx}\left[3^x\right]$ f) $\dfrac{d}{dx}\left[7^{2x-5}\right]$ g) $\dfrac{d}{dx}\left[9^{x^3}\right]$ h) $\dfrac{d}{dx}\left[5^{2x-x^2}\right]$

i) $\dfrac{d}{dx}\left[e^{\sin x}\right]$ j) $\dfrac{d}{dx}\left[4^{\tan x}\right]$ k) $\dfrac{d}{dx}\left[x^3 e^{4x}\right]$ l) $\dfrac{d}{dx}\left[\dfrac{e^x + e^{-x}}{e^x - e^{-x}}\right]$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/yg_497u6JnA`

iv) Derivative of logarithmic functions

This video shows how to find the derivative of the logarithmic functions listed.

a) $\dfrac{d}{dx}\left(\ln x\right)$ b) $\dfrac{d}{dx}\left(\ln x^2\right)$ c) $\dfrac{d}{dx}\left(\ln x^3\right)$

d) $\dfrac{d}{dx}\left(\ln(x+5)\right)$ e) $\dfrac{d}{dx}\left(\ln(x^2+4)\right)$ f) $\dfrac{d}{dx}\left(\ln(5+7x-x^3)\right)$

g) $\dfrac{d}{dx}\left(\ln(\sin x)\right)$ h) $\dfrac{d}{dx}\left(\ln\left(\sqrt[7]{x}\right)\right)$ i) $\dfrac{d}{dx}\left(\sqrt[3]{\ln x}\right)$

j) $\dfrac{d}{dx}\left(\log_3(x)\right)$ k) $\dfrac{d}{dx}\left(\log_7(5-2x)\right)$ l) $\dfrac{d}{dx}\left(\log_2(3x-x^4)\right)$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/Dp9sgIvaKPk`

v) Derivative of trigonometric functions

The video below shows how to find the derivative of the trigonometric functions listed below.

a) $f(x)=\sin(2x)$ b) $y=\cos(x^2)$ c) $y=x^2\cos x$

d) $f(x)=\cos^2 x$ e) $y=5\tan(3x)$ f) $y=\dfrac{\sin x}{x}$

g) $y=\dfrac{x^2\tan x}{\sin(2x)}$ h) $y=\sin^5(x^3+4x^2-2)$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/eEn6n6Odv1Q`

vi) Derivative of hyperbolic functions

The following video shows how to find the derivative of the hyperbolic functions listed below.

a) $\dfrac{d}{dx}\left[\sinh x\right]$ b) $\dfrac{d}{dx}\left[\sinh(4x)\right]$ c) $\dfrac{d}{dx}\left[\cosh(x^3)\right]$

d) $\dfrac{d}{dx}\left[\operatorname{sech}(x^4)\right]$ e) $\dfrac{d}{dx}\left[x^2\sinh x\right]$ f) $\dfrac{d}{dx}\left[\ln(\sinh(x))\right]$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/Q6-QZxUDfE0`

vii) Derivative of inverse trigonometric functions

The following video shows how to find the derivative of the inverse trigonometric functions listed below.

a) $f(x) = \sin^{-1}(5x)$ b) $y = \sin^{-1}(e^{6t})$

c) $g(x) = \sec^{-1}(e^x)$ d) $y = \tan^{-1}(x^2)$

e) $y = (\tan^{-1} x)^2$ f) $y = 2x^4 \sin^{-1}(x)$

g) $y = \tan^{-1}\left(\sqrt{5x}\right)$ h) $y = x\sin^{-1} x + \sqrt{1-x^2}$

i) $h(x) = (\arcsin x)\ln x$ j) $y = -\cos^{-1}\left(\dfrac{6x+3}{9}\right)$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/J2HmfIvDVCo`

viii) Derivative of inverse hyperbolic functions

The following video shows how to find the derivative of the inverse hyperbolic functions listed below.

a) $\dfrac{d}{dx}\left(\sinh^{-1}(4x)\right)$ b) $\dfrac{d}{dx}\left(\cosh^{-1}(x^3)\right)$

c) $\dfrac{d}{dx}\left(\tanh^{-1}(\sqrt{x})\right)$ d) $\dfrac{d}{dx}\left[\csc^{-1}(x)\right]^3$

e) Find $\dfrac{dy}{dx}$ for $y = 6x\sinh^{-1}(3x) - 2\sqrt{1+9x^2}$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/NQGiOuohNqI`

2.5 L'Hopital's Rule

The video linked below demonstrates how to apply the L'Hˆopital rule to find the limit of the following problems.

(a) $\lim_{x\to\infty} \frac{x^2}{e^x}$ (b) $\lim_{x\to\infty} \frac{\ln x}{x}$ (c) $\lim_{x\to 0} \frac{\sin(7x)}{\sin(4x)}$

(d) $\lim_{x\to 0} \frac{\sin(8x)}{3x}$ (e) $\lim_{x\to\infty} x \ln x$ (f) $\lim_{x\to\infty} \frac{1}{x^x}$

(g) $\lim_{x\to 0} [1-2x]^{\frac{1}{x}}$

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/Gh48aOvWcxw`

2.6 Tangents and Normals

The video in the link below demonstrates the concept of finding the tangent to a curve.

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/ktOYbZ8CpLA`

i) The video in the link below demonstrates the concept of finding tangent and normal equations using the curve with the equation $y = x^2 + 1$ at $x = 3$.

Video Visit the URL below to view a video:

`https://www.youtube.com/embed/9pWiu7VxXP4`

ii) The following video demonstrates how to find the tangent equations for the functions listed below at given points.

Video Visit the URL below to view a video:

https://www.youtube.com/embed/9pWiu7VxXP4

(a) $y = x^2 - x + 3$ at $(5,\ 15)$

(b) $y = x^3 - 2x^2 + 10$ at $(1,\ 9)$

(c) $y = x(x^2 - 5)$ at $x = 3$

2.7 Related Rate Word Problems

The videos in the links below demonstrate how to solve the following examples related to related rates.

Example 1: The side of a cube is increasing at the rate of $5cm^{-1}$. Find the rate of increase of the volume when the length of a side is $3cm$.

Solution: **Video** Visit the URL below to view a video:

https://www.youtube.com/embed/BucbEKBz2dY

Example 2: A water tank (inverted cone) with a radius of $2m$ and a height of $4m$. If water is pumped in at a rate of $2m^3$ per minute, find the rate at which the water level rises when the water is $3m$ deep.

Solution: **Video** Visit the URL below to view a video:

https://www.youtube.com/embed/iK625WaKomM

2.8 Graph Sketching by Investigating Stationary Points

The following videos demonstrate how to sketch curves by examining stationary points and their nature, as well as identifying intervals where the function is increasing or decreasing, for the examples listed below.

Example 1: Sketch the curve with the equation $y = x^3 + 3x^2 - 9x - 4$.

Solution: **Video** Visit the URL below to view a video:

https://www.youtube.com/embed/H-XDX7T0ADw

Example 2: Sketch the curve with the equation $f(x) = 3x^4 - 8x^3$.

Solution: **Video** Visit the URL below to view a video:

https://www.youtube.com/embed/sdCdxXzms7M